Discrete Mathematics

Quiz # 4 (April, 2015)

Name:____________________________ID:________________________________

1. (30%) Using strong induction to prove that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0, 2^1, 2^2, \ldots,\ 2^n \ldots$ and so on.

Let P(n) be the statement "n can be written as a sum of distinct powers of two"

Basis step: P(1) is true because $2^0 = 1$

Inductive step:

Assume P(i) is true $\forall i \le k, i \in N$.

If k+1 is even, $\frac{k+1}{2}$ ***is an integer smaller than k, and*** $P\left(\frac{k+1}{2}\right)$ ***is true. Therefore,***

P(k+1) is also true because k+1 can be written as $2 * \left(sum\ of\ \frac{k+1}{2}\right)$.

If k+1 is odd, it can be written as $2^0 + 2 * \left(sum\ of\ \frac{k}{2}\right)$***, which means P(k+1) is true.***

This completes the inductive step.

2. (30%) For a full binary tree T, $n(T)$ and $h(T)$ denote the number of vertices and the height of T, respectively. $h(T) = 0$ when T consists of only one vertex (i.e, root). Show that $n(T) = 2^{h(T)+1} - 1$ by using structural induction when the set of full binary trees is defined recursively by the following two steps:

   Basis step: A single vertex is a full binary tree.

   Recursive step: If T_1 and T_2 are disjoint full binary trees with $h(T_1) = h(T_2)$, $T_1 \cdot T_2$ is also a full binary tree, where $T_1 \cdot T_2$ represents a binary tree consisting of a root *r* together with the edges connecting *r* to each of the roots of the left subtree T_1 and the right subtree T_2.

Basis step:

For the full binary tree consisting of just a root, the result is true because n(T)=1 and h(T)=0, and $1 = 2^{0+1} - 1$.

Inductive step:

Assume that $n(T_1) = 2^{h(T_1)+1} - 1, n(T_2) = 2^{h(T_2)+1} - 1$ ***where*** $h(T_1) = h(T_2)$.

Then $n(T) = 1 + n(T_1) + n(T_2) = 1 + 2^{h(T_1)+1} - 1 + 2^{h(T_2)+1} - 1$

$= 2^{h(T_1)+1+1} - 1 = 2^{h(T)+1} - 1$

3. (40%) Use a recursive algorithm to represent each of the following functions.
   (a) $f(n, m) = n! \bmod m$, where $n, m \in \mathbb{N}$.
   (b) $g(a, n) = a^{2^n}$, where $a \in \mathbb{R}$ and $n \in \mathbb{N}$.

(a) Procedure modfactorial(n, m: positive integers)
If n=1 then modfactorial(n, m):=1
Else modfactorial(n, m):= (n*modfactorial(n-1, m)) mod m

(b) Procedure a(n: positive integer)
If n=1 then $\boldsymbol{a(n) \coloneqq a^2}$
Else $\boldsymbol{a(n) \coloneqq (a(n-1))^2}$